Analysis of the Total Number of Twists Resulting from Cutting any Order Moebius Bands With any Number of Cuts

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Introduction

Consider Figure 1.

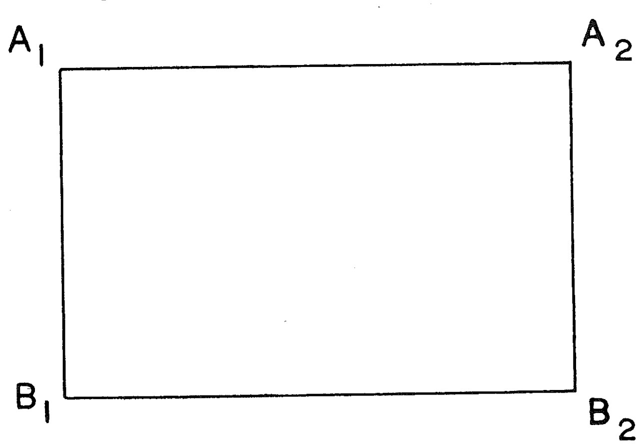

Figure 1.

If one end of this rectangle is given k half-twists[1] and then joined to the other end, a Moebius band of order k is formed.[2] In all odd order bands

[1] One half-twist = 180° rotattion. In the text of this report, I will use the terms *half-twist* and *twist* indiscriminately. When I mean a full twist (a 360° rotation) I will explicitly say *full twist.*

[2] Arnold, Bradford Henry, *Intuitive Concepts in lementary Topology* (Englewood Cliffs, N.J.: Prentice-Hall, 1962).

A_1 is joined to B_2 and B_1 to A_2, whereas in all even order bands A_1 is joined to A_2 and B_1 is joined to B_2 (Fig. 2).

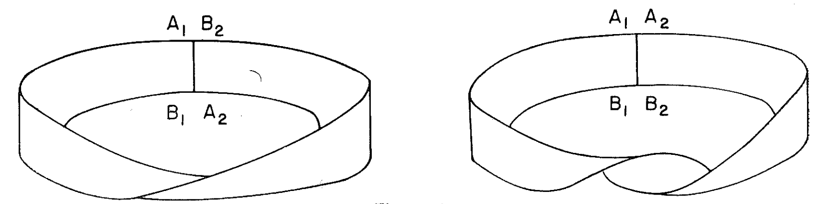

Figure 2.

The odd order bands are characterized by having only one side and one edge, i.e., any two points may be joined without crossing a boundary of the figure. An even order band, however, is not unilateral and is topologically equivalent to a circular right cylinder.

An interesting phenomenon associated with the Moebius band occurs when a band is cut lengthwise (Fig. 3). For example, if a band of order 1 is cut with 1 cut, a single band of 4 half-twists results, rather than the anticipated two bands of 1 twist each. When this band in turn is cut, two interlocking bands of 4 twists each are produced. In general, when a Moebius band is cut, the result is either a higher order band, or several linked bands of varying orders. The one exception is the trivial case of a band of order zero, in which the resulting bands are always unlinked bands of zero twists.

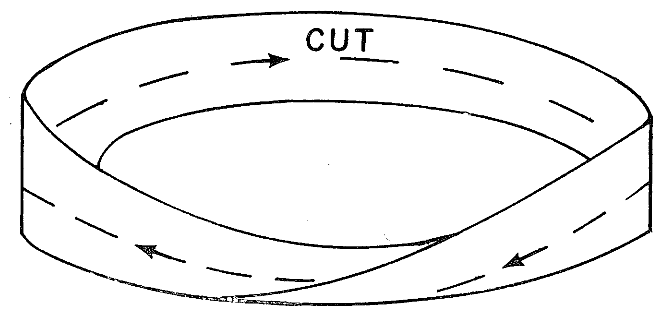

Figure 3. Moebius band of order 1, with the path of cut indicated.

That the result of cutting a Moebius band is indeed some sort of Moebius band or bands, and not some other figure is intuitivly obvious and is verified by a diagramatic representation of the cutting process (Fig. 4).

This paper is concerned with the *number* of twists resulting when any order band is cut with any number of cuts. Strictly speaking, this is not a topological consideration, because "twist" is a topological characteristic only so far as it determines evenness or oddness. Although topologically there is no difference between a band of order 1 and 7, geometrically there is.

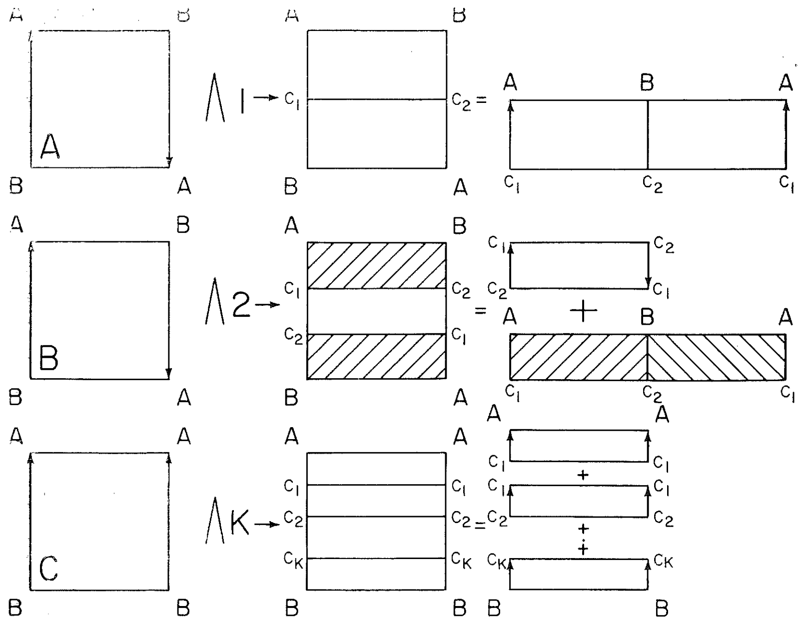

Figure 4. Diagramatic representation of the cutting process: A, an odd order is cut once, and an even order results; B, an odd order is cut twice, and two disjoint bands result; all other number of cuts with an odd order band are just combinations of 4(a) and 4(b); C, an even order band is cut by k cuts, and (k + 1) bands of the same order result. Note: $\wedge$ indicates the operation of cutting

This particular aspect of the Moebius band, namely number of twists, has been largely ignored because the Moebius band has been considered solely a topological problem. To my knowledge, no analysis relating to the number of twists resulting from cutting any order band has been made. Thus, the formula developed in this study is the first of its nature, as far as I know.

Definitions

1. Cut—A single cut is formed by piercing the band half way between the "edges" of the band, and cutting around the figure until the starting point is met (Fig. 3). In general, k cuts are formed by piercing the band at k points equally spaced along the perpendicular to one edge of the band, and cutting these parallel to the edge until each cut joins a starting point. In an even order band a given cut will always join its own starting point, whereas in an odd order band, this is not the case except for the middle cut (Fig. 5).
2. N_t—number of half-twists.
3. N_c—number of cuts.
4. $\wedge$—indicates the operation of cutting
5. $N_t \wedge N_c$—indicates sum of the total number of half-twists in the Moebius bands that are formed when a band of N_t half-twists is cut N_c number of times.

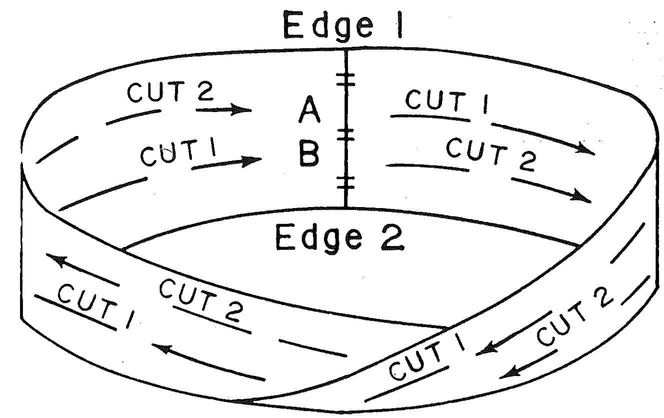

Figure 5. K = 2

6. Coherence of twists—any given band may be formed by twisting the band in one of two directions. One of these directions subtracts, and the other adds twists to the number already present. Two twists are said to be coherent if they are formed by twisting in a like manner (i.e., added to each other). The statement, "If two twists are coherent with a third twist, they are coherent with each other," follows immediately from the definition itself.

Introduction to the Proof

The basic equation to be verified is:

$$N_t \wedge N_c = N_tN_c + \{(N_c + \tfrac{1}{2}[1 - (-1)^{N_c}])\,(\tfrac{1}{2}[1 - (-1)^{N_t}])\}$$

An inductive proof is used, based on the following theorems or inductive formulae:

Theorem (1):

If N_t is *odd*, and $N_c \geq 2$, then
$N_t \wedge N_c = [(N_t \wedge 2) - N_t] + N_t \wedge (N_c - 2)$.

Theorem (2):

If N_t is *even*, and $N_c \geq 1$, then
$N_t \wedge N_c = [(N_t \wedge 1) - N_t] + N_t \wedge (N_c - 1)$.

Theorem (3):

$N_t \wedge N_c = (N_t - 2) \wedge N_c + 2 \wedge N_c$ (provided that $N_t \geq 2$, for $N_t - 2$ has no meaning if $N_t < 2$).

Theorems (1) and (2) are used to induct on the number of cuts and theorem (3) on the number of twists.

Proof of Theorems

Theorem (1):

If N_t is odd, and $N_c \geq 2$, then
$N_t \wedge N_c = [(N_t \wedge 2) - N_t] + N_t \wedge (N_c - 2)$.

Consider one section of any odd order band to be cut by N_c cuts (Fig. 6). If cut 1 and cut N_c are cut completely around the figure, thus cutting the band by 2, N_c will join with 1, and 1 with N_c. The total number of twists in the system is now $N_t \wedge 2$, consisting of the shaded and unshaded portions of Fig. 6.

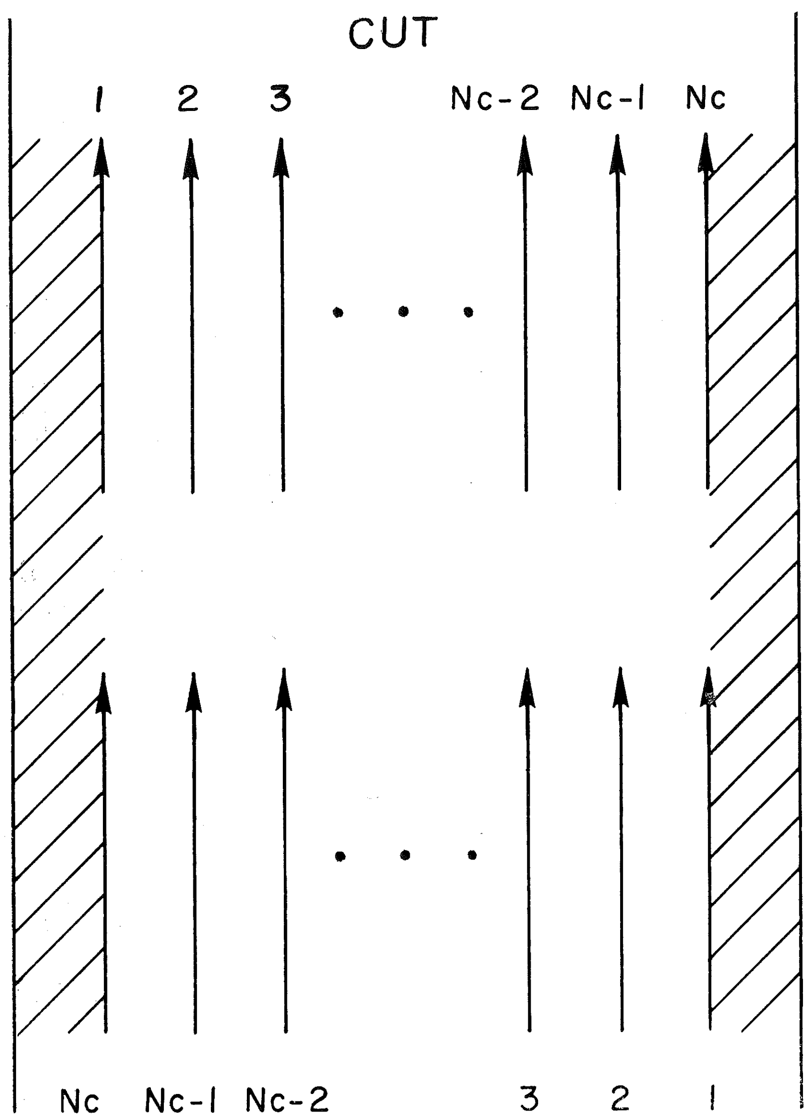

Figure 6.

As illustrated in Fig. 4(b), the shaded area is disjoint, though interlinked with the unshaded area. The unshaded area contains the original number of twists, N_t, and since the entire system contains $N_t \wedge 2$ twists, the shaded area must contain $(N_t \wedge 2) - N_t$ twists, (i.e., total number minus the number in the unshaded area).

The unshaded area is still to be cut by $N_c - 2$ more cuts. Since each pair of remaining cuts, (cut 2 and cut $N_c - 1$; cut 3 and cut $N_c - 2$; etc.), remain completely in the unshaded area, there is no interaction between them and cuts 1 and N_c.

Thus $N_t \wedge N_c =$ shaded area + unshaded area $\wedge (N_c - 2)$.

But the shaded area $= (N_t \wedge 2) - N_t$, and the unshaded area $= N_t$,

$$\therefore N_t \wedge N_c = [(N_t \wedge 2) - N_t] + N_t \wedge (N_c - 2).$$

The proof for theorem (2) is analagous, except that cut 1 meets itself when cut completely around the band, and thus the number of cuts may be reduced by 1 instead of 2 as follows:

$$N_t \wedge N_c = [(N_t \wedge 1) - N_t] + N_t \wedge (N_c - 1).$$

Theorem (3):

$$N_t \wedge N_c = (N_t - 2) \wedge N_c + 2 \wedge N_c, \ (N_t \geq 2).$$

To begin with, examine the case in which N_t is always even, (i.e., $N_t = 2n$). Theorem (3) then reads:

$2n \wedge N_c = (2n - 2) \wedge N_c + 2 \wedge N_c$, and shall be referred to as heorem (3a).

The case $n = 1$ is trivial because $(2 \cdot 1) \wedge N_c$ obviously equals $0 \wedge N_c + 2 \wedge N_c$. The truth for $n = 2$ is proven by inducting on N_c in theorem (3a).

When $n = 2$, theorem (3a) appears as:

$$(2 \cdot 2) \wedge N_c = 2 \wedge N_c + 2 \wedge N_c.$$

For $N_c = 1$,

$$(2 \cdot 2) \wedge 1 = 8 = 2 \wedge 1 + 2 \wedge 1. \text{ (Empirically true)}$$

Assume truth for $N_c = K$:

$$(2 \cdot 2) \wedge K = 2 \wedge K + 2 \wedge K;$$

Then the proof for $N_c = (K + 1)$ proceeds as follows:

$$(2 \cdot 2) \wedge (K+1) = [(2 \cdot 2) \wedge 1) - (2 \cdot 2)] + (2 \cdot 2) \wedge [(K+1) - 1] \quad \text{Theorem (2)}$$

$$= [(4 \wedge 1) - 4] + (2 \cdot 2) \wedge K$$

But, $[(4 \wedge 1) - 4] = [8 - 4] = 4$; (empirical observation)

And, $(2 \cdot 2) \wedge K = 2 \wedge K + 2 \wedge K$, (inductive hypothesis)

$$= 2\,(2 \wedge K);$$

$\therefore\ (2 \cdot 2) \wedge (K + 1) = 4 + 2\,(2 \wedge K),$
$= 2\,(2 + 2 \wedge K).$

Now, $2[2 \wedge (K + 1)] = 2\,\{[(2 \wedge 1) - 2] + 2 \wedge K\}$, (Theorem 2)
$= 2\,\{[(4) - 2] + 2 \wedge K\},$
(empirical observation)
$= 2(2 + 2 \wedge K),$

i.e., $2(2 + 2 \wedge K) = 2[2 \wedge (K + 1)]$.

But, $(2 \cdot 2) \wedge (K + 1) = 2(2 + 2 \wedge K),$

$\therefore\ (2 \cdot 2) \wedge (K + 1) = 2[2 \wedge (K + 1)]$
$= 2 \wedge (K + 1) + 2 \wedge (K + 1).$

Therefore, theorem (3a) is true for all N_c when $n = 2$. The proof for $n = 3$ is accomplished in an identical manner. Thus, theorem (3a) holds true for $n = 1, 2, 3$.

The proof of the equation in general will be based on the inductive procedure: "every non-empty set of positive integers has a smallest member." The set under consideration is the set of all n such that (3a) is *false*. It will be proven that this set has no smallest member and thus is empty, by showing that whenever (3a) is true for $n - 1$, it is true for n.

Since (3a) is true for $n = 1, 2, 3$, the smallest n for which it could be false is $n \geq 4$. If n is the smallest value for which (3a) is false, then $n - 1$, $n - 2$, . . ., must satisfy the equation. The proof that n (number of full-twists, not half-twists) must be true if $n - 1$ is true proceeds as follows:

Take a band of n coherent full-twists, n being the smallest value for which (3a) is false, and n thus ≥ 4. "Pinch off," or isolate two full twists, one at each end of the band, so that the system appears as digramtically represented in Fig. 7.

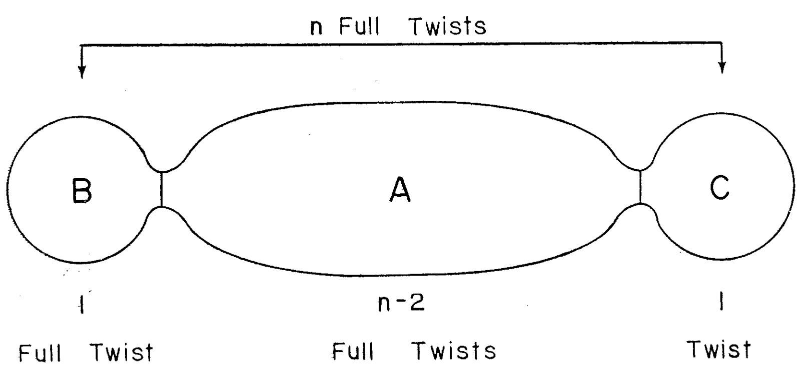

Figure 7. If n is given as the smallest value for which (3a) is false (thus $n \geq 4$) then if $n \wedge N_c = A \wedge N_c + B \wedge N_c + C \wedge N_c$, the set of false n is empty, and thus (3a) would be universally true.

The question now exists what happens when this system is cut completely around with N_c cuts. If the twists resulting from cutting B (i.e., $1 \wedge N_c$), and the twists resulting from cutting C (i.e., $1 \wedge N_c$), both add to the twists resulting from cutting A (i.e., $(n - 2) \wedge N_c$), this is essentially the same as saying $n \wedge N_c = (n - 2) \wedge N_c + 2 \wedge N_c$, since $1 \wedge N_c + 1 \wedge N_c = 2 \wedge N_c$ (see page ..... N.B. Here we are speaking in terms of whole-twists whereas on page .... in half-twists).

The proof that B and C in fact do add to A is simple: If section C of the system is ignored, and sections A and B cut with N_c cuts, the twists resulting from cutting B *must* add to A because this is the case for $n - 1$, i.e.,

$(n - 1) \wedge N_c = (n - 2) \wedge N_c + 1 \wedge N_c = A \wedge N_c + B \wedge N_c$

which is necessarily true since the inductive hypothesis is that n is the smallest value for which (3a) is false. Similarly, C and A will add when cut jointly while B is ignored.

Now, since the twists in sections B and C are both coherent with A, they are coherent with each other, and thus won't cancel each other out; and since each individually adds to A, they must therefore jointly add to A.

Thus $N \wedge N_c = A \wedge N_c + B \wedge N_c + C \wedge N_c,$
$= (n - 2) \wedge N_c + 1 \wedge N_c + 1 \wedge N_c$
$= (n - 2) \wedge N_c + 2 \wedge N_c.$

Since $(n - 1) \wedge N_c = (n - 2) \wedge N_c + 1 \wedge N_c,$
$(n - 2) \wedge N_c = (n - 1) \wedge N_c - 1 \wedge N_c.$

$\therefore\ n \wedge N_c = (n - 2) \wedge N_c + 2 \wedge N_c$
$= [(n - 1) \wedge N_c - (1 \wedge N_c)] + 2 \wedge N_c$
$= (n - 1) \wedge N_c + 1 \wedge N_c.$

i.e., $2n \wedge N_c = (2n - 2) \wedge N_c + 2 \wedge N_c.$

Therefore theorem (3a) is true for n; therefore, the set of n such that (3a) is false, is empty; therefore, (3a) is universally true.

The proof for theorem (3) for an odd number of twists proceeds in an analagous fashion.

With the inductive theorems established, the proof of the formula:

$N_t \wedge N_c = N_t N_c + N_t + \{(N_c + \tfrac{1}{2}[1 - (-1)^{N_c}])(\tfrac{1}{2}[1 - (-1)^{N_t}])\}$

will easily follow.

The equation is broken down into three cases as follows:

(I)	Even	$N_t \wedge N_c$	ie., $2m \wedge n = 2mn + 2m$
(II)	Odd	$N_t \wedge$ even N_c	i.e., $2m - 1) \wedge 2n = 4mn\ 2m - 1$
(III)	Odd	$N_t \wedge$ odd N_c	ie., $(2m - 1) \wedge (2n - 1) = 4mn$

where m and n are positive integers, relating to twists and cuts respectively.

The inductive formulae previously described will serve as the basis for the development of the proofs. Additionally, in each proof, one empirical observation (either $2 \wedge 1 = 4$; $1 \wedge 2 = 5$; or $1 \wedge 1 = 4$) is necessary to establish the equations for all m and n.

Proof of Cases

CASE I: Even $N_t \wedge N_c$, i.e., $2m \wedge n = 2mn + 2m$

Part 1. Let m be fixed, $m = 1$, then Case I becomes

$2 \wedge n = 2n + 2.$

For $n = 1$,

$2 \wedge 1 = 4$, (empirically true) $= 2 \cdot 1 \cdot 1 + 2 \cdot 1 = 2mn + 2m.$

Assume the truth for $n = K$:

$2 \wedge K = 2K + 2.$

Then the proof for $n = (K + 1)$ proceeds as follows:

$$2 \wedge (K+1) = [(2 \wedge 1) - 2] + 2 \wedge K \quad \text{(theorem 2)}$$
$$= [(4) - 2] + 2 \wedge K \quad \text{(empirical observation)}$$
$$= 2 + 2 \wedge K$$

But, $2 \wedge K = 2K + 2$ (inductive hypothesis)

$$\therefore\ 2 \wedge (K+1) = 2+2K+2 = 2K+4 = 2 \cdot 1 \cdot (K+1) + 2 \cdot 1$$
$$= 2mn + 2m$$

Therefore, by induction the equation is correct for all n if $m = 1$; i.e., (1,n) always satisfy Case I.

Part 2. Suppose the formula is *not* universally true, then there is a pair (m,n) of integers such that it is false. There is, by the principle previously used, a smallest m for which there is a smallest n such that Case I is false.

To verify Case I, it suffices to show that the set of (m,n) such that Case I is false is empty, by showing that, in fact, there is *no* smallest m for which there is a smallest n such that Case I is false.

If the smallest *m* for which there is a smallest *n* such that Case I is false is denoted by (m_1,n_1), the equation must be true for the pair $((m_1 - 1), n_1)$, i.e., $2(m_1 - 1) \wedge n_1 = 2(m_1 - 1)n_1 + 2(m_1 - 1) = 2m_1n_1 - 2n_1 + 2m_1 - 2.$

Examine Case I using the pair $(m_1,n_1$: $2m_1 \wedge n_1$.

By thereom 3: $2m_1 \wedge n_1 = (2m_1 - 2) \wedge n_1 + 2 \wedge n_1$;

but $(2m_1 - 2) \wedge n_1 = 2(m_1 - 1) \wedge n_1 = 2m_1n_1 - 2n_1 + 2m_1 - 2,$

and $2 \wedge n_1 = 2n_1 + 2$, as proven in Part of Case I.

$$\therefore\ 2m_1 \wedge n_1 = (2m_1n_1 - 2n_1 + 2m_1 - 2) + (2n_1 + 2)$$
$$= 2m_1n_1 + 2m_1.$$

This is identical to the equation for Case I, therefore, the pair (m_1,n_1) satisfies the equation. Thus, you cannot find an m for which there exists an n making Case I false. Therefore, the set of false values for Case I is empty, and thus it is universally true.

CASE II: Odd $N_t \wedge$ even N_c, i.e., $(2m - 1) \wedge 2n = 4mn + 2m - 1$

Part 1. Let m be fixed, $m = 1$, then Case II becomes

$1 \wedge 2n = 4n + 1.$

For $n = 1$,

$1 \wedge 2 = 5$ (empirically) $= 4 \cdot 1 \cdot 1 + 2 \cdot 1 - 1 = 4mm + 2m - 1.$

Assume truth for $n = K$:

$1 \wedge 2K = 4K + 1;$

Then the proof for $n = (K + 1)$ proceeds as follows:

$$1 \wedge 2(K + 1) = [(1 \wedge 2) - 1] + 1 \wedge 2K, \quad \text{(theorem 1).}$$

But $[(1 \wedge 2) - 1] = [5 - 1] = 4,$ (empirically true);

And $1 \wedge 2K = 4K + 1,$ (inductive hypothesis);

$$\therefore\ 1 \wedge 2(K + 1) = 4 + 4K + 1 = 4K + 5$$
$$= 4\,(K + 1) + 1.$$

Therefore, by induction the equation is correct for all n if $m = 1$.

Part 2. The proof for the equation in general proceeds as in Part 2 of Case I. The same notation will be used: (m_1,n_1) for the "smallest m for which there is an n such that the equation is false." That m_1 cannot be one is established in Part 1. Also it cannot be zero, for if it were, $(m_1 - 1)$ would be meaningless.

By a similar analysis as previously used, $((m_1 - 1), n_1)$, must satisfy Case II:

$$\text{i.e.: } [2(m_1 - 1) - 1] \wedge 2n_1 = 4(m_1 - 1)n_1 + 2(m_1 - 1) - 1$$
$$= 4m_1n_1 - 4n_1 + 2m_1 - 3$$

By theorem (3):

$(2m_1 - 1) \wedge 2n_1 = (2m_1 - 3) \wedge 2n_1 + 2 \wedge 2n_1;$

but $(2m_1 - 3) \wedge 2n_1 = [2(m_1 - 1) - 1] \wedge 2n_1$

$= 4m_1n_1 - 4n_1 + 2m_1 - 3,$ (inductive hypothesis)

and $2 \wedge 2n_1 = 4n_1 + 2$ by Case I.

$$\therefore\ (2m_1 - 1) \wedge 2n_1 = (4m_1n_1 - 4n_1 + 2m_1 - 3) + (4n_1 + 2)$$
$$= 4m_1n_1 + 2m_1 - 1\,.$$

This is identical to the equation for Case II using (m_1, n_1); therefore, the pair $(m_1\,n_1)$ satisfies the equation. Thus the set of false values for Case II is empty; thus Case II is universally true.

CASE III: Odd $N_1 \wedge$ odd N_c, i.e., $(2m-1) \wedge (2n-1) = 4mn$

Part 1. Let me be fixed, $m = 1$, then Case III becomes

$$1 \wedge (2n-1) = 4n.$$

For $n = 1$,

$$1 \wedge 1 = 4 \text{ (empirically true) } = 4 \cdot 1 \cdot 1 = 4mn.$$

Assume the truth for $n = K$:

$$1 \wedge (2K-1) = 4K.$$

Then the proof for $n = (K+1)$ proceeds as follows:

$$1 \wedge [2(K+1)-1] = 1 \wedge (2K+1)$$
$$= [(1 \wedge 2) - 1] + 1 \wedge (2K-1); \quad \text{(theorem 1)}$$

but, $[(1 \wedge 2) - 1] = 4$, (empirical observation)

and $[1 \wedge (2K-1)] = 4K$ (inductive hypothesis)

$$\therefore\ 1 \wedge [2(K+1)-1] = 4K + 4 = 4(K+1) = 4mn.$$

Therefore, by induction the equation is correct for all n if $m = 1$.

Part 2. The proof for all (m, n) follows the same pattern as in Cases I and II. Again (mn_1, n_1) has the same meaning as before; thus, the pair $((m_1-1), n_1)$ must satisfy the equation,

i.e. $[2(m_1-1)-1] \wedge (3n_1-1) = 4(m_1+1)n_1 = 4m_1n_1 - 4n_1$.

By theorem (3):

$$(2m_1-1) \wedge (2n_1-1) = (2m_1-3) \wedge (2n_1-1) + 2 \wedge (2n_1-1).$$

But $(2m_1-3) \wedge (2n_1-1) = [2(m_1-1)-1] \wedge (2n_1-1)$
$$= 4m_1n_1 - 4n_1.$$

And $(2 \cdot 1) \wedge (2n_1-1) = 2 \cdot 1 \cdot (2n_1-1) + 2 \cdot 1$ (Case I)
$$= 4n_1 - 2 + 2 = 4n_1.$$

$$\therefore\ (2m_1-1) \wedge (2n_1-1) = (4m_1n_1 - 4n_1) + 4n_1 = 4m_1n_1.$$

This is equivalent to the equation for Case III, therefore (m_1, n_1) satisfies the equation, and consequently, the set of false values for Case III is also empty, thus making the equation universally true.

Thus, through induction, Cases I, II and III are all universally true because the set of false values for the equations is empty, and consequently the combined equation:

$$N_t \wedge N_c = N_t N_c + N_t + \{(N_c + \tfrac{1}{2}[1-(-1)^{N_c}])(\tfrac{1}{2}[1-(-1)^{N_t}])\}$$

must be universally true.

Second Form of the Equation

A more informative form of the above equation would be one in which not only the total number of twists resulting from cutting would appear, but also the number of bands of each number of twists. For example: $1 \wedge 5 = 13 = 3$ bands of 4 twists each, and 1 band of 1 twist. As a direct result of the proofs of Theorems 1 and 2, the following form of the above equation yields this information. The coefficients of the terms in the square brackets are the number of bands of the number of twists indicated within the square brackets.

$$N_t \wedge N_c = (N_c+1)(\tfrac{1}{2})(1+(-1)^{N_t})\left[N_t\right] +$$

$$\frac{(N_c+1)}{2}(\tfrac{1}{4})(1-(-1)^{N_t})(1-(-1)^{N_c})\left[N_t \wedge 1\right] +$$

$$\frac{(N_c)}{2}(\tfrac{1}{4})(1-(-1)^{N_t})(1+(-1)^{N_c})\left[N_t \wedge 1\right] +$$

$$(1)(\tfrac{1}{4})(1-(-1)^{N_t})(1+(-1)^{N_c}\left[N_t\right]$$

where

$$N_t \wedge 1 = 2N_t + 2(1-(-1)^{N_t})$$

The proof proceeds trivially from the proofs of Thoerems 1 and 2. Examine the case odd $N_t \wedge$ odd N_c. As indicated in the proof of Theorem 1, cut 1 joins cut N_c, cut 2 joins cut $N_c - 1$, etc. Since there is an odd number of cuts, the last cut joins itself. After only the first cut and the last cut are joined, the system contains a total of $(N_t \wedge 1) + N_t$ twists, consisting of one band of $(N_t \wedge 1)$ twists and one band of N_t twists. All additional cuts, as demonstrated in the proof of Therem 1, take place in the single band of N_t twists. When the second cut joins the next to the last cut, there is a total of two bands of $(N_t \wedge 1)$ twists and one of N_t twists. Continuing in this fashion, when all the *pairs* of cuts have been joined, the system contains $\frac{N_c - 1}{2}$ bands of $(N_t \wedge 1)$ twists and one band of N_t twists. But this single band is yet to be cut one more time and thus the system contains $\frac{N_c - 1}{2}$ bands of $(N_t \wedge 1)$ twists plus one band of $(N_t \wedge 1)$ twists, or a total of $\frac{N_c + 1}{2}$ bands of $N_t \wedge 1$ twists. This appears in the above equation as

$$\frac{(N_c+1)}{2}(\tfrac{1}{4})(1-(-1)^{N_t})(1-(-1)^{N_c})\left[N_t \wedge 1\right].$$

The expression $(1/4)(1 - (-1)^{N_t})(1 - (-1)^{N_c})$ serves merely to regulate this term of the equation for the eveness or oddness of N_t and N_c. The proof for the case odd $N_t \wedge$ even N_c is identical, except that all the cuts are paired and thus there remains a single band of N_t twists. The proof of the even $N_t \wedge N_c$ case follows in the similar fashion from Theorem 2.

Acknowledgements

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